Problems

Exercise 1

Let AB and CD be two perpendicular diameters of a circle with centre O. Consider a point M on the diameter AB, different from A and B. The line CM cuts the circle again at N. The tangent at N to the circle and the perpendicular at M to AM intersect at P. Show that OP = CM.

Exercise 2

Let a, b, c be three non-zero integers. It is known that the sums $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$ are integers. Find these sums.

Exercise 3

For a real number x let [x] be the greatest integer less than or equal to x and let <x> = x – [x].

If a, b, c are distinct real numbers, prove that

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-a)(b-c)}+\frac{c^3}{(c-a)(c-b)}$$ is an integer if and only if <a> + <b> + <c> is an integer.

Exercise 4

For every positive integer k let a(k) be the largest integer such that $2^{a(k)}$ divides k. For every positive integer n determine a(1)+ a(2)+ a(3)+…+ a(2^n).

Exercise 5

In how many ways can the integers from 1 to 2006 be divided into three non-empty disjoint sets so that none of these sets contains a pair of consecutive integers?

Exercise 6

Let ABC be a right angled triangle at A. Denote D the foot of the altitude through A and O_1, O_2 the incentres of triangles ADB and ADC. The circle with centre A and radius AD cuts AB in K and AC in L. Show that O_1, O_2, K and L are on a line.